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AEC Computing Facility

New York, New York

NOTES ON MAGNETO-HYDRODYNAMICS - NUMBER I

GENERAL FLUID EQUATIONS

by

Harold Grad

AEC Computing Facility
Institute of Mathematical Sciences
New York University

Contents

Preface

This is the first in a series of notes on various topics in magneto-hydrodynamics. A large part of this material has been presented in informal lectures at irregular intervals since 1954. A set of four mimeographed notes, representing some of the early developments, was issued in July 1954 with the purpose of acquainting a small group at NYU with certain aspects of the subject. Meanwhile, a certain amount of general interest has been exhibited in these notes, and it was felt to be worthwhile to rewrite the original notes in a somewhat more readable form, at the same time extending and elaborating them greatly. The original four notes are contained in numbers 1, 3, 4, 7 of this series. Although many of these reports are unconnected, an attempt will be made to supply enough background material to make the series to some degree self-contained.

The status of the subject of magneto-hydrodynamics is somewhat paradoxical. It involves in combination, various aspects of fluid dynamics, electromagnetic theory, kinetic theory, and occasionally relativity. Most of this was available to Maxwell and he had the material to develop the subject to a considerable extent, but only recently has it been attacked significantly. Another, and probably related, consideration is that the companion

subjects of fluid dynamics and electromagnetism have a well developed experimental background, while relatively little is known experimentally, in magneto-hydrodynamics. Consequently, the development of magneto-hydrodynamic theory is rather abstract, principally devoted to the solution of rather idealized special problems, and the directions of development are guided to a large extent by what has been done in fluid dynamics and electromagnetics. There do exist important problems in astrophysics, in properties of the earth's atmosphere, and in gaseous discharges, but these are usually so difficult as to be unapproachable in the framework of a mathematical theory. Controlled experiments, with only a few effects present at a time, are very scarce.

This first note is largely a matter of bookkeeping and can be motivated only by pointing to the various applications made later on or by reference to kinetic theory. Various forms of fluid conservation equations are listed for future reference. They are presented in forms too general to be of specific mathematical use, but they do serve the useful purpose of correlating and categorizing the many specific fluid descriptions which will be used.

This note was prepared with the assistance of Ignace Kolodner and Albert Blank.

General Fluid Equations

Any macroscopic picture of a fluid, electrically conducting or not, is based on conservation laws, usually of mass, momentum, and energy, and occasionally of others. It is important to realize several things. First, these are "lack of conservation" more than "conservation" equations. For example, the energy of an element of fluid (or of the whole fluid) is usually not a constant, but is assumed to vary in certain definite ways. Second, the form taken by these equations is, macroscopically speaking, pure postulation. One is governed by certain general considerations of what has been found to be useful, but no question of derivation can arise unless referred to a more primitive basis such as motion of particles. This remark applies also to the thermo-dynamic formalism which is microscopically derivable but macroscopically postulated. Finally, in a strictly formal sense, many of the terms included below are redundant. For example, in the momentum equations, we find terms X^i and $\partial P^{ij}/\partial x^j$. Clearly, it would be completely general to omit $\partial P^{ij}/\partial x^j$ and claim that the force term X^i sometimes takes the form of space derivatives of auxiliary quantities, and, conversely, it is easy to see that a force term X^i can always be put into the form $\partial P^{ij}/\partial x^i$ by suitable choice of P^{ij}.

This separation into terms is significant only because the conservation equations are incomplete, mathematically speaking, and the supplementary relations which will eventually have to be introduced will be of several kinds for which the various forms such as $\partial P^{ij}/\partial x^j$ and X^i are more convenient. It may be remarked that, even from a molecular standpoint, the separation into $\partial P^{ij}/\partial x_j$ and X^i turns out to be somewhat arbitrary. Insofar as the equations listed below are sufficiently general to cover most cases of normal occurrence, they cannot be complete but must be supplemented by relations which specify the particular nature of the fluid under study; these relations will range from functional relations (e.g., an equation of state) to differential equations (e.g., relating stresses to velocity gradients, or relating a force term to a magnetic field vector which in turn is coupled to other of the fluid variables through a set of auxiliary electromagnetic differential equations).

By the inclusion of general terms like X_r, we obtain complete and universal validity for these equations (e.g., by defining X_r as the force required to make the momentum equation correct!), but, by the same token, they cannot serve more than a bookkeeping purpose. This use is important, however, because of the great variety of fluid models which may be used to describe a conducting medium.

1. Notation

We consider a fluid consisting of n homogeneous components, each with its own intrinsic properties (such as molecular mass, charge, etc.) and each with its own thermodynamic state variables.

Molecular properties:

m_r = mass

ε_r = charge (signed) $r = 1, \ldots, n$

$\gamma_r = \varepsilon_r / m_r$

Molecular structure is referred to only to establish the connection between mass density and charge density below.

State variables:

ν_r = number density

$\rho_r = m_r \nu_r$ = mass density

$q_r = \varepsilon_r \nu_r$ = charge density (signed)

$q_r / \rho_r = \gamma_r$ = constant for a given fluid

u_r^i = macroscopic flow velocity

$\rho_r u_r^i$ = momentum per unit volume

e_r = internal energy per unit volume

$\bar{e}_r$ = total energy per unit volume = $e_r + \frac{1}{2} \rho_r u_r^2$

T_r = temperature

p_r = pressure

μ_r = chemical potential

Supplementary variables:

P_r^{ij} = stress tensor, $p_r^{ij} = P_r^{ij} - p_r \delta^{ij}$

Q_r^i = heat flow

X_r^i = force per unit volume

$\mathcal{E}_r$ = energy source per unit volume

σ_r = mass source (chemical creation) per unit volume

$\sigma_r \omega_r^i$ = momentum source (chemical) per unit volume

Internal energy, stresses, etc., are defined to be those observed moving with the individual fluid, i.e., at the velocity u_r.

2. Conservation Equations

The following "conservation" equations are postulated:[*]

$$(1) \begin{cases} \dfrac{\partial \rho_r}{\partial t} + \dfrac{\partial}{\partial x^j}(\rho_r u_r^j) = \sigma_r & \text{(mass)} \\[2em] \dfrac{\partial}{\partial t}(\rho_r u_r^i) + \dfrac{\partial}{\partial x^j}(\rho_r u_r^i u_r^j + P_r^{ij}) = X_r^i + \sigma_r \omega_r^i & \text{(momentum)} \\[2em] \dfrac{\partial \bar{e}_r}{\partial t} + \dfrac{\partial}{\partial x^j}(\bar{e}_r u_r^j + u_r^i P_r^{ij} + Q_r^j) = \mathcal{E}_r & \text{(energy)} \end{cases}$$

Using $v_r = u_r - u$, $\sum \rho_r v_r = 0$ we define the following gross (single-fluid) variables:

$\nu = \sum \nu_r$

$\rho = \sum \rho_r$

$q = \sum q_r$

$\rho u = \sum \rho_r u_r$

[*] The summation convention is used for repeated tensorial indices, but not for indices distinguishing the fluids.

$$e = \sum (e_r + 1/2\, \rho_r u_r^2) - 1/2\, \rho u^2 = \sum (e_r + 1/2\, \rho_r v_r^2)$$

$$\bar{e} = \sum \bar{e}_r = e + 1/2\, \rho u^2$$

$$X = \sum X_r$$

$$\mathcal{E} = \sum \mathcal{E}_r$$

$$P^{ij} = \sum (\rho_r u_r^i u_r^j + P_r^{ij}) - \rho u^i u^j = \sum (P_r^{ij} + \rho_r v_r^i v_r^{\,j})$$

$$p = \sum (p_r + \tfrac{1}{3}\rho_r v_r^2), \quad p^{ij} = P^{ij} - p\delta^{ij}$$

$$Q^i = \sum (\bar{e}_r u_r^i + u_r^j P_r^{ij} + Q_r^i) - (\bar{e}u^i + u^j P^{ij})$$

$$= \sum (Q_r^i + \bar{e}_r v_r^i + v_r^j P^{ij} - \rho_r v_r^i u^j v_r^j)$$

The v_r are the relative or diffusion velocities. The
single fluid internal energy, stresses, etc., are defined
with respect to the mass flow velocity, u, and are not
merely sums of the corresponding quantities of the
constituent fluids.

From the conservation of mass and an appropriate
definition of ω_r,

$$\sum \sigma_r = 0$$
$$\sum \sigma_r \omega_r = 0.$$

Adding the individual conservation equations:

$$(2) \quad \begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x^j} (\rho u^j) = 0 \\[2mm] \dfrac{\partial}{\partial t} (\rho u^i) + \dfrac{\partial}{\partial x^j} (\rho u^i u^j + P^{ij}) = X^i \\[2mm] \dfrac{\partial \bar{e}}{\partial t} + \dfrac{\partial}{\partial x^j} (\bar{e}u^j + u^i P^{ij} + Q^j) = \mathcal{E} \end{cases}$$

It is sometimes convenient to consider the following (Lagrangian) form of the momentum equations:

$$(3) \quad \rho_r \left[\frac{\partial u_r^i}{\partial t} + u_r^j \frac{\partial u_r^i}{\partial x^j} \right] + \frac{\partial P_r^{ij}}{\partial x^j} = X_r^i + \sigma_r \, (\omega_r^i - u_r^i)$$

$$(4) \quad \rho \left[\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} \right] + \frac{\partial P^{ij}}{\partial x^j} = X^i$$

and also the difference momentum equations

$$(5) \quad \frac{\partial v_r^i}{\partial t} + u_r^j \frac{\partial u_r^i}{\partial x^j} - u^j \frac{\partial u^i}{\partial x^j} + \frac{1}{\rho_r} \frac{\partial P_r^{ij}}{\partial x^j} - \frac{1}{\rho} \frac{\partial P^{ij}}{\partial x^j}$$

$$= \frac{X_r^i}{\rho_r} - \frac{X^i}{\rho} + \frac{\sigma_r}{\rho_r} \, (\omega_r^i - u_r^i)$$

Equations (3) and (4) are obtained by combining the momentum and mass equations of (1) and (2) respectively.

The conservation equations (2) are equivalent (assuming differentiability of all quantities) to the integral relations

$$(6) \quad \begin{cases} \dfrac{d}{dt} \displaystyle\int_D \rho dV + \oint_S \rho u^j dS_j = 0 \\[2em] \dfrac{d}{dt} \displaystyle\int_D \rho u^i dV + \oint_S (\rho u^i u^j + P^{ij}) dS_j = \int_D X^i dV \\[2em] \dfrac{d}{dt} \displaystyle\int_D \bar{e} dV + \oint_S (\bar{e} u^j + u^i P^{ij} + Q^j) dS_j = \int_D \mathcal{E} dV \end{cases}$$

The surface integrals are taken over the boundary, S, of the fixed domain, D. The Lagrangian form of (6) for a domain D(t) which is carried along by the fluid at the velocity u is

$$(7)\begin{cases} \dfrac{d}{dt} \int\limits_{D(t)} \rho dV = 0 \\[2em] \dfrac{d}{dt} \int\limits_{D(t)} \rho u^i dV + \oint\limits_{S(t)} P^{ij} dS_j = \int\limits_{D(t)} X^i dV \\[2em] \dfrac{d}{dt} \int\limits_{D(t)} \bar{e} dV + \oint\limits_{S(t)} (u^i P^{ij} + Q^j) dS_j = \int\limits_{D(t)} \mathcal{E} dV \end{cases}$$

Similar integral forms hold for the individual conservation equations (1).

3. Charge and Current

We note the conservation of charge

$$\sum \gamma_r \sigma_r = 0$$

and define the current densities

$$J_u = \sum q_r v_r, \quad J = \sum q_r u_r = J_u + qu;$$

J is the current observed in the fixed system and J_u "moving with the fluid"; the convection current is qu. From (1), the individual "conservation" of charge is

$$(8) \qquad \frac{\partial q_r}{\partial t} + \frac{\partial}{\partial x^j} (q_r u_r^j) = \gamma_r \sigma_r$$

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whence, adding,

(9) $\qquad \frac{\partial q}{\partial t} + \text{div } J = 0.$

Similarly, from (1) we can obtain individual current equations

(10) $\qquad \frac{\partial}{\partial t}(q_r u_r^i) + \frac{\partial}{\partial x^j}(q_r u_r^i u_r^j + \gamma_r P_r^{ij}) = \gamma_r(X_r^i + \sigma_r \omega_r^i).$

and the total current equation

(11) $\qquad \frac{\partial J^i}{\partial t} + \sum_r \frac{\partial}{\partial x^j}(q_r u_r^i u_r^j + \gamma_r P_r^{ij}) = \sum_r \gamma_r(X_r^i + \sigma_r \omega_r^i).$

From this and

(12) $\qquad \frac{\partial}{\partial t}(qu^i) + u^i \frac{\partial J_u^j}{\partial x^j} + \frac{\partial}{\partial x^j}(qu^i u^j) + \frac{q}{\rho}\frac{\partial P^{ij}}{\partial x^i} = \frac{q}{\rho} X^i$

follows

(13) $\qquad \frac{\partial J_u^i}{\partial t} + \frac{\partial}{\partial x^j}(J_u^i u^j) + J_u^j \frac{\partial u^i}{\partial x^j} + \frac{\partial}{\partial x^j}(\sum q_r v_r^i v_r^j)$

$\qquad\qquad + \sum \gamma_r \frac{\partial P^{ij}}{\partial x^j} = \frac{q}{\rho}\frac{\partial P^{ij}}{\partial x^j} = \sum \gamma_r (X_r^i + \sigma_r \omega_r^i) - \frac{q}{\rho} X^i.$

The current equation, (11) or (13), is particularly significant in case there are just two fluids, one positive and one negative. In this case, the mass velocity u and current J are as complete a description as the two velocities u_1 and u_2, and the total momentum equation and current equation can be used to replace the two momentum equations. Equation (13) can be written in terms of J_u and u alone using the relation

(13a) $\qquad \sum q_r v_r^i v_r^j = \frac{\gamma_1 \rho_2 + \gamma_2 \rho_1}{(\gamma_1 - \gamma_2)^2 \rho_1 \rho_2} J_u^i J_u^j$

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4. Entropy

A local thermodynamic structure can be introduced by making certain assumptions. First (this has already been done), extensive quantities such as mass, momentum, and energy are introduced as densities. We know from equilibrium thermodynamics that all thermodynamic parameters can be computed from the single function "entropy density" as a function of the other densities. We need only adopt the same functional relation in nonequilibrium and compute all other quantities, e.g., temperature and pressure, by differentiation.

We define the entropy per unit volume, s, by

$$(14) \qquad Ts = e + \sum \mu_r \rho_r + p$$

and write the second law

$$(15) \qquad Tds = de + \sum \mu_r d\rho_r$$

and the Gibbs-Duhem relation

$$0 = ed(\tfrac{1}{T}) + \sum \rho_r d(\tfrac{\mu_r}{T}) + d(\tfrac{p}{T}).$$

We have expressed the intensive thermodynamic coordinates per unit volume rather than per unit mass since this provides a simpler (additive) transition from individual fluid properties to total fluid properties. Expressed in terms of unit mass, (14) and (15) take the form

$$(16) \qquad \left\{ T\,\frac{s}{\rho} = \frac{e}{\rho} + \sum \mu_r \frac{\rho_r}{\rho} + \frac{p}{\rho} \right.$$

(16) $\qquad \left\{ Td(\frac{s}{\rho}) = d(\frac{e}{\rho}) + \sum \mu_r d(\frac{\rho_r}{\rho}) + pd(\frac{1}{\rho}). \right.$

From the above we compute that

(17) $\qquad \frac{\partial s}{\partial t} + \text{div}\ (su + \frac{Q}{T} + \frac{1}{T} \sum \mu_r \rho_r v_r) = \frac{1}{T}\ (\mathcal{E} - u \cdot X + \sum \mu_r \sigma_r)$

$\qquad\qquad + Q \cdot \text{grad}(\frac{1}{T}) + \sum \rho_r v_r \cdot \text{grad}\ (\frac{\mu_r}{T}) - \frac{1}{T} p^{ij} \frac{\partial u^i}{\partial x^j}\ .$

   This thermodynamic formalism is appropriate in case
it is sufficient to deal with the total momentum and
energy equations alone, although taking n individual mass
equations. In case we wish to treat all n velocities and
all n temperatures as primitive variables, it is necessary
to introduce the generalized n fluid formalism:*

(18) $\qquad \begin{cases} T_r s_r = e_r + \mu_r \rho_r + p_r \\[2mm] T_r ds_r = de_r + \mu_r d\rho_r \\[2mm] 0 = e_r d(\frac{1}{T_r}) + \rho_r d(\frac{\mu_r}{T_r}) + d(\frac{p_r}{T_r}) \end{cases}$

The total entropy per volume is

(19) $\qquad s* = \sum s_r$

We compute that

(20) $\qquad \frac{\partial s*}{\partial t} + \text{div}\ (\sum s_r u_r + \sum \frac{Q_r}{T_r}) = \sum \frac{1}{T_r}\ (\mathcal{E}_r - u_r \cdot X_r)$

$\qquad\qquad + \sum \frac{\sigma_r}{T_r}\ (\mu_r + \frac{1}{2}u_r^2 - u_r \cdot \omega_r) + \sum Q_r \cdot \text{grad}\ (\frac{1}{T_r})$

$\qquad\qquad - \sum \frac{1}{T_r} p_r^{ij} \frac{\partial u_r^i}{\partial x^j}\ .$

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* This is significant only for perfect gases, i.e.,
when there is no significant intercomponent energy.

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It is interesting to note that the combination $\mu_r + \frac{1}{2} u_r^2$
would have appeared in such a fashion as to warrant the
name chemical potential if we had chosen as thermodynamic
coordinates mass, total energy, and momentum instead of
mass and internal energy, (see Grad, J. Phys. Chem., 56,
1039-1048, 1952).

A few words should be said about the difference
between (17) and (20). First of all, the "entropy" s*
is not the same as s; specifically, it is not a function
of the same variables. In the one fluid formulation, (17),
there is an entropy source related to diffusion velocities
and gradients of the chemical potentials. This does not
appear in (20); in other words, the individual fluid
velocities are here treated as primitive variables, conse-
quently reversible. If the energy sources are related to
the forces by the relation $\mathcal{E}_r = u_r \cdot X_r$, then (20) considers
this as reversible, while (17) has an entropy source of
magnitude

$$\mathcal{E} - u \cdot X = \sum v_r \cdot X_r \; ;$$

i.e., slipping is irreversible. On the other hand, if we
have energy sources, $\mathcal{E}_r$, in the absence of forces, X_r,
and have $\mathcal{E} = \sum \mathcal{E}_r = 0$, then (17) considers this state of
affairs as reversible, while (20) does not if the various
fluids are at different temperatures.

The choice of the entropy function s* can be

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justified on the basis of statistical mechanics; however, its most important macroscopic justification is that we may keep the concept of increasing entropy together with a fluid description involving several momentum and energy equations.

As an example, choose for the force X_r and energy $\mathcal{E}_r$ the expressions

$$(21) \quad \begin{cases} X_r = q_r E + q_r u_r \times B \\ \mathcal{E}_r = u_r \cdot X_r = q_r u_r \cdot E \end{cases}$$

(E and B are the electromagnetic field vectors.) There is no entropy production from this source in the second case, (20), while in (17) we have

$$(22) \quad \mathcal{E} - u \cdot X = J_u \cdot E_u = (J - qu) \cdot (E + u \times B)$$

The entropy source is the product of the current and electric field each measured in a system moving with the instantaneous local fluid velocity, u.

Another possibility is

$$(23) \quad \begin{cases} X_r = q_r E + (q_r u_r + J_r) \times B \\ \mathcal{E}_r = E \cdot (q_r u_r + J_r) \end{cases}$$

The extra currents, J_r, could be introduced, for example, to take into account magnetic polarization. In this case, we have

$$(24) \quad \begin{cases} X = qE + J \times B \\ \mathcal{E} = E \cdot J \end{cases}$$

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by redefining

(25) $$J = \sum (q_r u_r + J_r)$$

(instead of just $\sum q_r u_r$), whence

$$\mathcal{E} - u \cdot X = E_u \cdot J_u$$

as before, but also

(26) $$\mathcal{E}_r - u_r \cdot X_r = (E + u_r \times B) \cdot J_r \, .$$

It will be observed that we have not included electric or magnetic polarization terms in the thermodynamics. Nevertheless, we are not restricted to the case of unit dielectric constant and permeability. To achieve this we need merely adopt the viewpoint ahat J and q include polarization current and charge respectively (cf. Stratton, equation (7), p. 12). Moreover, this interpretation is almost forced upon us in any consideration involving energy. The usual formulation in which electric energy is represented by $\int E \cdot dD$ rather than $\int \frac{1}{2} E^2$ (identical remarks hold for H and B) is a useful expedient only when the internal energy of the medium is disregarded. In certain very simple situations, it is convenient to distinguish electric charges which are under the direct control of the experimentor from other "polarization" charges. In the process of polarization, there is a transfer of energy between the electric field and the dielectric (internal thermal energy). If one is not interested in this transfer of energy (which

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is frequently reversible), one can maintain an artificial conservation of those energies which one chooses to acknowledge by using $\int E \cdot dD$ instead of $\int \frac{1}{2} E^2$. In oomplex problems of fluid flow, retention of D and H would seem to be a hindrance rather than a help. It should be remarked that equations (8) and (10) - (13) are valid only in the absence of polarization.

In succeeding notes chemical terms will be ignored unless the context plainly demands their inclusion.

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